Liquidity, Interest, and Asset Prices*

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Abstract

To study the effect of monetary policy and liquidity on the pricing of goods and assets, we introduce a transactions demand for money into a simple general equilibrium model.

Preliminary and incomplete.

1 Introduction

Recent events, in the

turbulence in the financial system inevitably raises questions about what can and should be done by policy makers. Should the Federal Reserve System target stock prices as well as the prices of goods and services in formulating its monetary policy? Can interest rates control the stock market? What impact does the wealth effect have on real consumption? What is the optimal response to financial crises? Can timely provision of liquidity prevent crises? Should crises be prevented at all costs or is a cost-benefit calculus required before intervention is warranted?

Among many other attempts to provide answers to these questions, Allen and Gale (1994, 1998, 2000a,b,c, 2002a,b) investigate the interrelationships between liquidity, asset markets, and financial crises. The models in these papers, like the rest of the literature, are limited in two respects. First, they essentially real (non-monetary) models. Secondly, they focus on banks and banking, to the exclusion of other parts of the financial system. This paper describes some strategies for incorporating money and monetary policy in a more satisfactory way in models of financial intermediation and financial markets.

A theme that runs through the series of papers referred to above is the importance of liquidity for the pricing of assets and the efficiency of risk sharing in financial markets . Liquidity, in the sense of immediate command over goods and services, is a commodity like any other. If markets are complete, the allocation of liquidity will be efficient. If markets are incomplete, the allocation of liquidity will be inefficient. Moreover, there is a link between inefficiency of market provision of liquidity, on the one hand, and asset-pricing volatility, incomplete risk sharing and costly financial crises, on the other.

To see this, consider the behavior of an individual who is uncertain about his future demand for liquidity (liquidity preference). In some states, he has a high liquidity preference, for example, because he wants to consume immediately; in other states, he has a low liquidity preference. If markets are complete, he can provide himself with the optimal amount of liquidity in each state. Since everyone faces the same state-contingent prices for liquidity, the allocation is efficient. This has an immediate implication for asset pricing. Since agents do not need to buy or sell assets from liquidity motives, asset prices are determined by future asset returns, independently of individual liquidity preferences.

When markets are incomplete, by contrast, liquidity plays a crucial role in the determination of asset prices. Suppose there are no markets for insuring against liquidity shocks. Individuals will provide for liquidity preference by holding assets. An investor who receives a high liquidity preference shock (increased need for liquidity) is forced to sell assets. If this happens on a large scale, asset prices will fall. This has three immediate consequences. First, asset prices are now determined partly by liquidity conditions, in addition to future returns. Secondly, investors are forced to sell assets at *low* prices when demand for liquidity is *high*. In other words, they are forced to take a loss in states where the marginal utility of income is high. Thus, the market provides adverse insurance, the opposite of what is needed for efficient risk sharing. Thirdly, the attempt to obtain liquidity by selling assets may prove self-defeating, as every fall in asset prices requires an even larger sale, which drives prices even lower. In extreme cases, institutions and investors may find themselves unable to meet their commitments, some will default, and a full blown crisis follows.

Allen and Gale (1994, 1998) model liquidity phenomena using real assets. The demand for liquidity is based on the Diamond and Dybvig (1983) model of liquidity preference. The supply of liquidity is determined by investment decisions made by individual investors or financial institutions. There are two broad classes of assets, short-term and long-term. Short-term assets are "liquid" in the sense that they produce a return quickly; long-term assets are "illiquid" because they only produce returns after the passage of a longer period of time. The "liquidity" of the market is measured not by the numbers of traders in a market but by the liquidity of their portfolios. A market is liquid, in this theory, if market participants collectively hold a large proportion of their wealth in the form of liquid assets. Even a very thick market, such as the New York Stock Exchange, can be illiquid if the participants hold very illiquid portfolios.

This framework can be used to illustrate the role of liquidity in determining asset price volatility, risk sharing, and financial crises. While individuals can obtain liquidity by selling assets, the amount of liquidity available to the market as a whole is limited by the stock of short-term (liquid) assets. Holding short-term assets is costly because the rate of return tends to be lower than the rate of return on long-term asset. To compensate the providers of liquidity for holding the short-term asset, there must be an extra return, which comes from the possibility of buying long-term assets cheaply when the demand for liquidity is high. So a fall in asset prices is needed to compensate liquidity providers and the supply of liquidity can never be great sufficient to prevent a fall in asset prices in equilibrium. Thus, in the absence of complete markets, the provision of liquidity is inefficient, asset-price volatility is high and financial crises may result.

In these models, for the most part, assets have real payoffs and liquidity consists of current consumption. In a true monetary model, by contrast, liquidity is identified with fiat money. Since fiat money is created by the CB at virtually zero cost, it is more or less a free good. Moreover, the real money supply is determined endogenously by the price level. A sufficiently steep fall in the price level will increase the real quantity of money to arbitrarily large levels. So, in a monetary model, how can there ever be a shortage of liquidity? How can liquidity prevent the existence of stable and efficient asset prices? In fact, the change from a real to a monetary model of liquidity raises some tricky issues, as we shall see.

Money matters in Allen and Gale (1998, 2000b,c) to the extent that contracts are denominated in terms of money as a unit of account. By controlling the price level or the exchange rate, the central bank (CB) can prevent financial crises and implement more efficient risk sharing. In effect, the CB is solving a security design problem, determining the ex post real payoffs corresponding to pre-specified ex ante nominal payoffs. However, the price level and the exchange rate in these models are purely notional parameters. There is no real function for money and no account of how control of the money supply or interest rates determines the price level within a theory of general equilibrium.

In order to show how the CB's control of money supply and interest rates affects asset prices and the real value of debt and equity, we first need to provide a non-trivial role for money. In Section 2, we address the determination of the price level in a model of pure exchange. A transactions demand for money is provided by introducing a cash-in-advance constraint (Clower, 1968). If the cash-in-advance constraint is binding for every agent, adding up the individual cash-in-advance constraints provides an aggregate constraint that looks very much like the Quantity Equation from classical monetary theory. In fact, it is a market-clearing condition that says that the money supply is equal to the aggregate demand for money. This additional equation is enough to determine one additional variable. However, the Quantity Equation is sufficient to determine the price level only if we make an implicit assumption that (a) the cash-in-advance constraints are binding and (b) the price level is known with certainty at the moment individuals choose how much money to hold. If either of these conditions is violated, the price level is indeterminate.

It is relatively easy to ensure that the cash-in-advance constraint is binding. For example, if liquidity is costly, individuals will economize on money balances and hold the minimum amount consistent with their planned transactions. This leaves us with the problem that a single equation cannot determine a random price level. The determination of the price level under uncertainty is discussed in Section **??**, where we show that the CB has other instruments it can use to control the price level. As long as the interest rate is positive (i.e., liquidity is costly), the CB earns seigniorage which it must spend in equilibrium. The CB's budget constraints in each state provide us with additional equilibrium conditions, one for each possible price level. If the CB's can commit to its demand for goods and supply of fiat money in each state, we have enough equations to determine the remaining equilibrium variables, including the value of money (the inverse of the price level).

One of the important questions that arises in this framework is whether we can legitimately regard the CB's actions as exogenous or whether it is better to regard them as (passively) accommodating the prices determined in the market. This is a question to which we don't at the moment have an answer.

Fluctuations in the price level clearly have important implications for the real economy whenever contracts or securities are denominated in terms of money. Inflation can reduce the real value of debts, and thus reduce the chance of default or a financial crisis, and conversely deflation will have opposite effects. One of the most important questions we focus on is how CB policy controls real liquidity and how this impacts asset prices. In Section 4 we re-interpret the model as a model of asset pricing and consider the implications of price-level uncertainty for the pricing and trading of assets. On the one hand, if liquidity is costless, fluctuations in nominal asset prices are innocuous (in the absence of nominal debts and other non-contingent commitments). Even if the money supply is fixed and exogenous, the price level can adjust to provide whatever level of liquidity is needed. On the other hand, if liquidity is costly, the cash-in-advance constraint must bind in some states, and that means that asset prices will be affected by the liquidity constraint, that is, the limited supply of costly liquidity. However, the main conclusion is that the direct impact of liquidity shocks is felt directly on the volume of assets traded and only indirectly on prices. Furthermore, while the CB's control of prices and interest rates affects *nominal* prices, it does not directly affect *real* asset prices, that is, the prices of assets relative to goods.

To examine real asset-price volatility, we change the model to separate the trading of assets and goods. There are separate cash-in-advance constraints and separate budget constraints in the asset market and goods market. Liquidity shocks in the asset market, in the presence of a fixed amount of liquidity, will change (nominal) asset prices relative to (nominal) goods prices. In fact, the CB's attempt to maintain a constant (goods) price level will exacerbate the real volatility of asset prices. To illustrate this, we consider a CB policy that targets a fixed (goods) price level and assume that asset demands are subject to individual liquidity shocks that cancel out in the aggregate. In the absence of liquidity constraints (or in the presence of complete markets), asset prices will not fluctuate. However, when markets are incomplete and the liquidity constraint binds, real asset prices are affected by liquidity shocks. Increases in trading volume raise the demand for liquidity which can only be met if asset prices fall. Thus, costly liquidity by itself may lead to asset-price volatility while the CB targets the price level.

In Section 5 we return to some of the issues with which we began and discuss the way in which these tools can be used to address other issues in the theory of intermediation and financial markets.

2 Money and the price level

In modern payments systems, access to liquidity is (virtually) free within the trading day, whereas obtaining liquidity overnight involves significant costs. Within the trading day, banks are able to run very large "daylight overdrafts" that enable them to bridge the gap between payments and receipts at minimal cost, whereas payments that are not covered at the end of the day must be balanced by overnight borrowing on the interbank market (Coleman, 2002; Martin, 2002; Zhou, 2000). In general, the cost of liquidity will depend on the purpose for which it is needed. The transactions demand for liquidity is very short term: in spot transactions payment and delivery are closely matched. The cost of liquidity for these kinds of transactions may be very low or negligible. On the other hand, the precautionary or speculative demand for liquidity is longer term. Financial transactions may involve holding or shorting assets over a long and perhaps uncertain period of time. Here the liquidity costs may be significant. In this section, we focus on the transactions demand for liquidity. Although the cost of liquidity is assumed to be small, it still plays an important role in the determination of the price level. Later, in Section 4, we will study transactions in asset markets, where the costs of liquidity may be higher.

It is well known that a change in the price level has real effects if securities and contracts are denominated in terms of money (Pigou, 1943). While the effect of money and the price level has been thoroughly studied in macroeconomics, less attention has been pay to the impact of money in financial economics, which tends to rely on "real" models. A prominent exception, which illustrates the importance of money and nominal prices in a financial context, is the model of general equilibrium with financial securities introduced by Cass (1984) and Werner (1985). Financial securities are claims that promise payment in an abstract unit of account contingent on the state of nature. Any change in the future price level changes the "real" returns of these securities. The Cass-Werner model in its original form has no account of how the price level is determined. The price level is, in fact, indeterminate and that gives rise to an indeterminacy of equilibrium. Moreover, the indeterminacy is not of the innocent kind found in classical models where only relative prices matter: the indeterminacy of the price level causes a real indeterminacy of equilibrium.

One way of resolving the indeterminacy of equilibrium is to introduce a transactions demand for money. Then the corresponding market-clearing condition (demand for money equals supply of money) gives an additional equation with which we can determine the price level. This is the approach adopted by Magill and Quinzi (1992), and Geanakoplos and Dubey (1992), who introduce a transactions demand for money based on the cash-in-advance constraint (Clower, 1968; Lucas, 1990; Fuerst, 1992).

Unfortunately, the market-clearing condition for money is sufficient to determine the price level only if the demand for money is well defined. If it is costless to hold excess money balances, the quantity of money no longer determines the price level, since any quantity of money can be absorbed by the system without affecting the demand and supply of goods. In a related vein, Magill and Quinzii point out that the price level in their model is determinate if money is held as medium of exchange in equilibrium, but not if it is held as a store of value. So our first observation is that, in order to use the demand for and supply of money to determine the price level, liquidity must be costly.

In the next section we present a simple model of costly liquidity. Agents can borrow as much liquidity as they want from the central bank, but they must pay interest on their borrowing. Because liquidity is costly, the cash-in-advance constraint is always binding. In the first model we look at, the binding cash-in-advance constraint is sufficient to determine the price level. Later on we shall see that even this may not be enough.

2.1 Transactions demand for money

Clower (1968) enunciated the axiom that in actual markets, unlike classical general equilibrium theory, "money buys goods and goods buy money". To capture this realistic feature of exchange, he added the now-famous cash-in-advance constraint to a model of pure exchange. We plan to follow Clower and then go further by adding a central bank (CB) and a cost of liquidity.

A *pure exchange economy* consists of I agents, indexed by $i = 1, ..., I$. The agents trade ℓ commodities and fiat money. Each agent i is characterized by a consumption set $\mathbf{R}_+^\ell$, an endowment $e_i \in \mathbf{R}_+^\ell$, and a utility function $u_i : \mathbf{R}_+^\ell \to \mathbf{R}$.

Exchange is normally assumed to take place at a single instant or date, but in order to model the process of monetary exchange more precisely, time is here divided into three sub-periods. Each sub-period represents a different stage in the trading process.

- In the first sub-period, agents borrow money from the CB.
- In the second sub-period, agents exchange goods for money and money for goods.
- In the final sub-period, agents repay their loans to the CB with interest.

The CB provides liquidity to the market so agents can carry out transactions. The CB earns interest on the loans it makes in the first sub-period and spends this income on goods in the second sub-period. Because the interest payments are received in the third sub-period after the goods are traded, the CB is forced to inject additional money into the economy in the second period. In fact, these injections are necessary in order for the agents to have enough money to repay their loans with interest.

Let $M > 0$ denote the amount of money supplied by the CB in the first sub-period and let $g \in \mathbf{R}_+^\ell$ and $\Delta M \geq 0$ denote the bundle of goods demand and the injection of money supplied in the second sub-period. We assume that the CB's choice M, g and ΔM is exogenous but has to satisfy a budget constraint. Alternative formulations of the CB's policy are discussed in Section **??**.

Agents require money in order to buy goods and so, in the first sub-period, each agent borrows an amount of money that will allow him to carry out the planned transactions in the second sub-period. The money he receives in exchange for selling goods must exceed the amount he originally borrowed so that he can repay his loan with interest in the final period. Let $m_i \geq 0$ denote the money balance agent i borrows from the CB in the first sub-period and let $x_i \in \mathbf{R}^{\ell}_{+}$ denote the bundle of goods demanded in the second sub-period. Let $p \in \mathbf{R}^{\ell}_{+}$ and $r \geq 0$ denote, respectively, the vector of money prices of goods and the interest rate on loans from the CB. The cash-in-advance constraint requires that the value of purchases be less than or equal to the agent's cash balance, that is,

$$p \cdot (x_i - e_i)^+ \leq m_i, \tag{1}$$

where for any vector $z = (z_1, ..., z_\ell)$, the notation z^+ denotes the vector of non-negative excess demands defined by

$$z^+ = (\max\{z_1, 0\}, ..., \max\{z_\ell, 0\}).$$

After trade in the second sub-period, the amount of money agent i holds is equal to his initial balance m_i minus the value of his net trades $p \cdot (x_i - e_i)$. In order to repay his loan with interest he needs $(1 + r)m_i$ units of money in the final sub-period. So agent i can repay his loan with interest if and only if his budget constraint

$$p \cdot (x_i - e_i) + rm_i \leq 0 \tag{2}$$

is satisfied. Agent i's behavior is summarized by a decision problem in which he chooses a money balance m_i and a commodity bundle x_i to maximize his utility subject to the budget constraint and the cash-in-advance constraint. Formally, he has to choose an ordered pair $(x_i, m_i) \in \mathbf{R}^{\ell}_{+} \times \mathbf{R}_{+}$ to maximize $u_i(x_i)$ subject to the cash-in-advance constraint (1) and the budget constraint (2).

Because of the usual homogeneity properties, there is no essential loss of generality in assuming the money supply M is fixed in the sequel, so the CB's policy choice can be summarized by $(g, \Delta M)$.

An allocation is an array $(x, m) = ((x_i, m_i))$ such that $(x_i, m_i) \in \mathbf{R}^{\ell}_{+} \times \mathbf{R}_{+}$ for each agent i. The allocation (x, m) is *attainable* if it satisfies the market-clearing conditions

$$\sum_{i=1}^{I} x_i + g = \sum_{i=1}^{I} e_i \tag{3}$$

and

$$\sum_{i=1}^{I} m_i = M. \tag{4}$$

The first condition (3) is the market-clearing condition for money in the first sub-period. The second condition (4) is the market-clearing condition for goods in the second sub-period.

We assume that the CB chooses its policy $(g, \Delta M)$ before the markets open and treat the policy $(g, \Delta M)$ as given and exogenous in defining an equilibrium. However, the CB has to anticipate the equilibrium to ensure that its budget constraint will be satisfied at the equilibrium prices and interest rate.

Given a policy $(g, \Delta M)$, an *equilibrium relative to the policy* $(g, \Delta M)$ is defined to consist of an attainable allocation (x, m) and a vector of prices and interest rate (p, r) such that, for every agent i, (x_i, m_i) solves

$$\begin{array}{ll} \max & u_i(x_i) \\ \text{s.t} & p \cdot (x_i - e_i)^+ \leq m_i, \\ & p \cdot (x_i - e_i) + r m_i \leq 0, \end{array}$$

and the CB's budget constraint

$$p \cdot g = \Delta M,$$

is satisfied. Note that attainability and the budget constraints imply that

$$rM = \Delta M$$

so the money supply in the final sub-period is just sufficient to repay the loans to the CB.

2.2 Price level determination

Liquidity, like any other valuable commodity, is costly. In practice, the opportunity cost of liquidity is determined by the alternative productive uses to which wealth in the form of liquid assets might be put. In the present model, we adopt the simplification of assuming the only cost of liquidity is the interest charged by the CB on loans. A more realistic model would contain alternative productive uses of the funds borrowed from the CB, which would ensure a positive opportunity cost. Costly liquidity in the form of a positive interest rate $r > 0$ forces agents to economize on liquid balances and this in turn helps determine the price level. The importance of a positive interest rate for price determination can be illustrated by considering the limiting case of an economy in which the CB earns no seigniorage, $(g, \Delta M) = (0, 0)$, and consequently the interest rate $r = 0$.

To see that the price level is indeterminate, let $(x, m, p, 0)$ be an equilibrium and consider what happens if all prices are reduced in the same proportion, that is, replace the price vector p with λp for some $0 < \lambda < 1$. The homogeneity of demand ensures that goods markets continue to clear at the new price vector λp. The cash-in-advance constraints, which were satisfied before, are now strictly satisfied, but that too is consistent with equilibrium because liquidity is costless. Thus, the price level can be reduced without affecting equilibrium in an essential way.

Moreover, if the cash-in-advance constraints are slack at the initial equilibrium $(x, m, p, 0)$, the price level can be increased without disturbing the equilibrium conditions. For example, the equilibrium vector p can be replaced by λp for any λ sufficiently close to 1, and λp will be an equilibrium price vector as well.

Proposition 1 *Let (x, m, p, r) be an equilibrium for the policy $(g, \Delta M) = (0, 0)$ that raises no seigniorage. Then $r = 0$ and $(x, m, \lambda p, 0)$ is an equilibrium for the same policy, for any $\lambda > 0$ such that*

$$\lambda p \cdot (x_i - e_i)^+ \leq m_i, i = 1, ..., I.$$

The proof of Proposition 1 makes it clear that the price level is indeterminate because the cash-in-advance constraints are not binding. To ensure that each agent's cash-in-advance constraint is binding, liquidity must be costly. If the interest rate r is positive, each agent will choose to hold the minimum money balance m_i that allows him to satisfy his cash-in-advance constraint. So a positive interest rate is sufficient for individual cash-in-advance constraints to hold as equations in equilibrium. Then summing the individual constraints yields an aggregate cash-in-advance constraint

$$\sum_{i=1}^{I} p \cdot (x_i - e_i)^+ = M. \tag{5}$$

We can think of equation (5) as a version of classical monetary theory's Quantity Equation. It gives us the additional equation we need to determine the equilibrium price level.

As was pointed out earlier, in a modern payments system, the intraday interest cost of liquidity is vanishingly small, so it is interesting to consider what happens as the CB's seigniorage and the associated interest rate both converge to zero. The next result shows that, in the limit, the equilibrium corresponds to a classical competitive equilibrium. However, since the individual cash-in-advance constraints hold as equations for any positive interest rate, they must also hold in the limit.

Theorem 2 *Let (x^n, m^n, p^n, r^n) be a sequence of equilibria corresponding to a sequence of policies $(g^n, \Delta M^n)$. Suppose that $(x^n, m^n, p^n, r^n) \to (x^0, m^0, p^0, r^0)$ as $(g^n, \Delta M^n) \to (0, 0)$. For each i, assume that u_i is continuous and locally non-satiable and*

$$p^0 \cdot e_i > 0.$$

Then (x^0, m^0, p^0, r^0) is an equilibrium for the policy $(0, 0)$ and

$$p^0 \cdot (x_i^0 - e_i)^+ = m_i^0$$

for $i = 1, ..., I$. Furthermore, (p^0, x^0) is a Walrasian equilibrium.

Proof. This follows from the usual upper hemi-continuity argument. ■

Thus, in contrast to the earlier examples of equilibria in the limit economy with $r = 0$, the limit of a sequence of equilibrium with $r > 0$ must satisfy the Quantity Equation, so we have an extra equation to determine the price level. The next result shows that if the cash-in-advance constraint is binding for each agent, then the equilibrium price level is determinate in the limit economy.

Theorem 3 DETERMINATENESS OF PRICE LEVEL. *Let (x, m, p, r) be an equilibrium for the policy $(g, \Delta M) = (0, 0)$ and suppose that for each agent $i = 1, ..., I$,*

$$p \cdot (x_i - e_i)^+ = m_i.$$

If the Walrasian equilibria of the exchange economy are locally unique, then the equilibrium (x, m, p, r) is locally unique among the set of equilibria with binding cash constraints.

Proof. Let (x, m, p, r) be an equilibrium relative to the policy $(g, \Delta M) = (0, 0)$ and suppose that the individual cash-in-advance constraints are all binding. If (x, m, p, r) is not locally unique among the set of equilibria with binding constraints, then for any $\varepsilon > 0$ we can find an equilibrium (x', m', p', r'), relative to the policy $(g, \Delta M) = (0, 0)$, ε-close to (x, m, p, r). Clearly, $r = r' = 0$ and (x, p) and (x', p') are Walrasian equilibria, so the local uniqueness of Walrasian equilibria implies that $x = x'$ and $p = kp'$ for some $k > 0$. Then the aggregate cash-in-advance constraint

$$\sum_{i=1}^{I} p \cdot (x_i - e_i)^+ = M = \sum_{i=1}^{I} p' \cdot (x'_i - e_i)^+$$

implies that $p = p'$ and the individual cash-in-advance constraints imply that $m_i = m'_i$ for each i. ■

To sum up the story so far, if liquidity is costly, then individual cash-in-advance constraints are binding, the aggregate cash-in-advance constraint plays the role of the Quantity Equation in classical monetary theory, and the money supply 'determines' the price level.

3 The price level under uncertainty

The problem with the argument presented in the preceding section is that it assumes the price level is known with certainty at the moment individual agents choose their money balances. The small cost of liquidity ensures that the cash-in-advance constraint is binding giving us one extra equation, and this one equation is enough to determine a single variable, the equilibrium price level. However, if there is uncertainty about the price level, a single equation will not suffice to determine the different price levels. Put another way, a small but positive cost of liquidity ensures that agents will minimize their cash balances so that the cash-in-advance constraint is binding in one state, but this is consistent with slackness in other states. Obviously, the cash-in-advance constraint cannot help to determine the price level in states in which it is slack (i.e., a strict inequality). Fortunately, there is another set of equilibrium relations that does serve to determine the price level: the CB supplies money in exchange for goods and this 'open market operation' determines the price level. We can conclude that the (stochastic) price level is determinate, but the theory depends on the active involvement of the CB and not on the cash-in-advance constraint (Quantity Equation).

To illustrate these ideas we extend the exchange economy to allow for (extrinsic) uncertainty. There is a finite set of states $s = 1, ..., S$ with a common-knowledge prior probability distribution $\pi = (\pi_1, ..., \pi_S) \gg 0$. In the first sub-period, agents know the distribution of the state. The true state is revealed at the beginning of the second sub-period, i.e., after agents have chosen their money demands and before trade in goods begins.

As before, we assume that the initial money supply $M > 0$ is fixed in the first sub-period and that the CB demands a bundle of goods $g(s) \in \mathbf{R}^{\ell}_{+}$ and issues a quantity of money ΔM in each state s in the second sub-period. Note that the uncertainty represented by the state of nature s is extrinsic in the sense that it does not affect preferences or endowments, but we do allow the CB's demand for goods $g(s)$ to depend on the state. However, the injection of money ΔM in the second sub-period is independent of s. This is in fact a requirement for equilibrium. Since the initial money supply M and the interest rate r are independent of s, the final demand for money to repay loans in the last sub-period $(1+r)M$ is also independent of s. Market clearing requires $rM = \Delta M$, so the injection of money in the middle sub-period must be independent of s also.

Agent i borrows $m_i \geq 0$ units of money from the CB in the first sub-period and demands a bundle of goods $x_i(s)$ in each state s in the second sub-period.

Let $p(s) \in \mathbf{R}^{\ell}_{+}$ denote the equilibrium vector of goods prices in state s. Then agent i satisfies the budget constraint

$$p(s) \cdot (x_i(s) - e_i) + rm_i \leq 0$$

and the cash-in-advance constraint

$$p(s) \cdot (x_i(s) - e_i)^{+} \leq m_i$$

in each state s in the second sub-period.

An allocation (x, m) is *attainable* if

$$\sum_{i=1}^{I} x_i(s) + g(s) = \sum_{i=1}^{I} e_i, \forall s,$$

and

$$\sum_{i=1}^{I} m_i = M.$$

For a given policy $(g, \Delta M)$, an *equilibrium relative to the policy* $(g, \Delta M)$ consists of an attainable allocation (x, m) and a price vector (p, r) such that , for every agent i, (x_i, m_i) solves

$$\begin{array}{ll} \max & \sum_{s=1}^{S} \pi_s u_i(x_i(s)) \\ \text{s.t} & p(s) \cdot (x_i(s) - e_i) + r m_i \leq 0, \forall s, \\ & p(s) \cdot (x_i(s) - e_i)^+ \leq m_i, \forall s. \end{array}$$

and the CB's policy satisfies the budget constraint

$$p(s) \cdot g(s) = \Delta M, \forall s.$$

The budget constraints together with the market-clearing conditions imply that $rM = \Delta M$ so there is the right amount of money to pay the loans at the last date.

If preferences are locally non-satiable, then $r > 0$ implies that, for each agent i,

$$\max_{s=1,\ldots,S} \left\{ p(s) \cdot (x_i(s) - e_i)^+ \right\} = m_i.$$

But this is not enough to determine the price level, even locally, as the following example shows.

Example 4 *Assume there are two agents $i = 1, 2$, two goods $h = 1, 2$, and endowments $e_1 = (2, 0)$ and $e_2 = (0, 2)$. The agents have identical Cobb-Douglas utility functions $u_i(x_i) = x_{i1}^{1/2} x_{i2}^{1/2}$. Suppose there are two states $s = 1, 2$ and let*

$$g(s) = \begin{cases} (0.2, 0.2) & s = 1, \\ (0.4, 0.4) & s = 2. \end{cases}$$

Then the prices satisfy $p(1) = 2p(2)$. The cash-in-advance constraints will be binding in the state with the high price level but not in the state with the low price level. To see this, note that agent i's excess demands will be

$$x_i(s) - e_i = \begin{cases} (0.9, 0.9) - e_i & s = 1, \\ (0.8, 0.9) - e_i & s = 2. \end{cases}$$

For agent 1, say, $(x_1(1) - e_1)^+ = (-1.1, 0.9)^+ = (0, 0.9)$ and $(x_1(2) - e_1)^+ = (-1.2, 0.8)^+ = (0, 0.8)$, so

$$\begin{aligned} p(1) \cdot (x_1(1) - e_1)^+ &= 0.9 p_2(1) \\ &= 1.8 p_2(2) \\ &> 0.8 p_2(2) \\ &= p(2) \cdot (x_1(2) - e_1)^+ . \end{aligned}$$

The calculation for agent 2 is symmetric. Summing the individual cash-in-advance constraints, we see that the Quantity Equation holds in state 1 but not in state 2. Thus, we can use the Quantity Equation to solve for $p(1)$ but not $p(2)$.

Nonetheless, the equilibrium price level is "determined" by the budget constraints

$$p(s) \cdot g(s) = \Delta M, \forall s, \tag{6}$$

which provide an additional equation for each state and price level.

This answers the question about the determinateness of the price level when $r > 0$, but not in the limit as $(g, \Delta M) \to (0,0)$. In the limit, when $g(s) = 0 = \Delta M$, both sides of equation (6) are identically zero and tell us nothing about the price level. Instead, we have to take limits and pay particular attention to the way in which the CB's policy approaches the limit. For concreteness, consider a sequence of equilibria $\{(x^n, m^n, p^n, r^n)\}$ relative to a corresponding sequence of policies $\{(g^n, \Delta M^n)\}$ and suppose the policies take the form

$$(g^n, \Delta M^n) = \frac{1}{n}(\gamma, \mu),$$

for each n. Then (6) reduces to

$$p^n(s) \cdot \gamma(s) = \mu, \forall s,$$

for each n and each state s and, in the limit, as $(x^n, m^n, p^n, r^n) \to (x^0, m^0, p^0, r^0)$, we have S additional equations,

$$p^0(s) \cdot \gamma(s) = \mu, \forall s,$$

which suffice to determine the price level.

Theorem 5 *Let $\{(x^n, m^n, p^n)\}$ be a sequence of equilibria relative to the sequence of corresponding policies $\{(g^n, \Delta M^n)\} = \left\{\frac{1}{n}(\gamma, \mu)\right\}$. Suppose that $(x^n, m^n, p^n, r^n) \to (x^0, m^0, p^0, r^0)$ as $n \to \infty$. For each i assume that u_i is continuous and locally non-satiable and*

$$p^0(s) \cdot e_i > 0.$$

Then (x^0, m^0, p^0, r^0) is an equilibrium for the policy $(g^0, \Delta M^0) = (0,0)$ and

$$p^0(s) \cdot \gamma(s) = \mu, \forall s. \tag{7}$$

Furthermore, $(p^0(s), x^0(s))$ *is a Walrasian equilibrium for each* $s = 1, ..., S$. *If the Walrasian equilibria of the exchange economy are locally unique, then the equilibrium* (x^0, m^0, p^0, r^0) *is locally unique among the set of equilibria in which the "budget constraints" (7) are satisfied.*

Proof. The proof is similar to Theorem 3. ∎

Returning to the numerical example above, we can illustrate how the budget constraints suffice to determine the equilibrium price levels.

Example 6 *Use the same parameter values as in Example 6, but let the CB's demand for goods be* $g^n = \frac{1}{n}\gamma$ *where*

$$\gamma(s) = \begin{cases} (0.2, 0.2) & s = 1, \\ (0.4, 0.4) & s = 2, \end{cases}$$

and let $\Delta M = \frac{1}{n}\mu$. *Then for each* n, $p^n(s) \cdot \gamma(s) = \mu$ *so in the limit* $p^0(s) \cdot \gamma(s) = \mu$ *for each* s. *When there is no seigniorage,* $x_i(s) = (1, 1)$ *for each* i *and* s, *so the equilibrium prices must be proportional to* $(1, 1)$. *Then we can solve the seigniorage equation for the limiting prices*

$$p^0(s) = \begin{cases} \left(\frac{\mu}{0.2}, \frac{\mu}{0.2}\right) & s = 1, \\ \left(\frac{\mu}{0.4}, \frac{\mu}{0.4}\right) & s = 2. \end{cases}$$

Thus, control of the money supply and real seigniorage allows the CB to control the price level. Even in the limit, where there is no seigniorage, the possibility of small amounts of seigniorage allows to explain how the price level is determined by the CB's demand decisions. The crucial assumption, of course, is that CB policy is exogenous to the model. If, as we shall discuss in the next section, the CB adopts an accommodating policy, with seigniorage accommodating changes in the price level, the indeterminacy would be reinstated.

3.1 Alternative specifications of CB policy

We have assumed that the CB chooses a policy $(g, \Delta M)$ independently of the prices (p, r). On the one hand, since the CB is a large player that can influence endogenous variables like the price level, it makes sense to treat the CB as a Stackleberg 'leader'. On the other hand, the CB does face a budget constraint so one might want to insist that the CB make its choices conditional on (p, r). One possible formulation would be to make the demand for goods and supply of money functions of the spot prices, so that the budget constraint,

$$p(s) \cdot g(p(s), r, s) \equiv \Delta M(p(s), r, s),$$

is identically satisfied in each state s. In order to have an equilibrium, it is still necessary that

$$\Delta M(p(s), r, s) = rM$$

so putting $g(p(s), r, s)$ equal to a constant vector $\gamma(s)$ gives us exactly the same mathematical structure as before. It has to be admitted that the determinateness of the price level is being imposed by brute force: once it is assumed that the CB can dictate the level of seigniorage to be collected, everything else follows of necessity.

3.2 Complete markets

It may be thought that incomplete markets have something to do with the indeterminateness of the price level. But we can introduce Arrow securities without changing these results. In fact, Arrow securities will ensure $x(s)$ is independent of s but as the last theorem shows even when the allocation is Walrasian there can be price level "indeterminacy" in the limit.

In modeling Arrow securities, we must be careful to specify whether they affect only the budget constraint or the budget constraint and the cash-in-advance constraint.Suppose for example that Arrow securities promising one unit of money in sub-period 2 are traded in sub-period 1 and, for simplicity, suppose further that they are not subject to the cash-in-advance constraint in sub-period 1. If $q(s)$ denotes the price of one unit of money in state s, then the budget constraint of individual i is

$$\sum_{s \in S} q(s)p(s) \cdot (x_i(s) - e_i) + rm_i \leq 0.$$

What is the appropriate cash-in-advance constraint? One possibility is that purchases must be paid for and short bond positions must be simultaneously paid for in cash. In that case, the cash-in-advance constraint in state s is

$$p(s) \cdot (x_i(s) - e_i)^+ + z_i(s)^+ \leq m_i.$$

The significance of this formulation is that while Arrow securities allow redistribution of consumption across states, they do not allow redistribution of liquidity. The determination of the price level is essentially the same as in the case of incomplete markets.

Note that there are two types of incompleteness: due to liquidity cost and due to unhedged risk. If $r > 0$ one does not necessarily get efficient risk sharing even with a full set of Arrow securities.

3.3 Existence

Given a Walrasian equilibrium corresponding to the policy $(g, \Delta M) = (0, 0)$, existence of equilibrium corresponding to a policy $(g, \Delta M)$ in some neighborhood of $(0, 0)$ (i.e., in some neighborhood of a Walrasian equilibrium) follows from the implicit function theorem under the usual conditions plus the assumption that $x_{ih} \neq e_{ih}$ for every good h and every agent i.

Let (p^*, x^*) be a Walrasian equilibrium such that $p^* \gg 0$ and $x_i^* \gg 0$ for every i. Suppose further that $x_{ih}^* \neq e_{ih}$ for $h = 1, ..., \ell$ and every i. Let $H_i = \{h : x_{ih}^* > e_{ih}\}$. If u_i is strictly quasi-concave then, for any (p, r) in some neighborhood N_i of $(p^*, 0)$, there is a unique solution $f_i(p, r)$ to the problem of maximizing $u_i(x_i)$ subject to the budget constraint $\pi_i(p, r) \cdot (x_i - e_i) \leq 0$, where

$$\pi_{ih}(p, r) \equiv \begin{cases} p_h & \text{if } h \notin H_i, \\ (1 + r)p_h & \text{if } h \in H_i. \end{cases}$$

Assume that f_i is C^1 on N_i. Then consider the system of equations

$$\sum_{i=1}^{I} f_i(p, r) + g = \sum_{i=1}^{I} e_i.$$

For every g sufficiently small, there is at least one solution (p^g, r^g) to these equations, and we can clearly choose M^g and ΔM^g so that (p^g, r^g) is an equilibrium relative to $(g, M^g, \Delta M^g)$.

4 Asset price volatility

So far, we have focused on the case where $r = 0$ because this corresponds most closely to the classical Walrasian equilibrium and because it seems the appropriate assumption for modeling daily transactions executed through a modern payments system. A longer time-frame is appropriate when we consider liquidity in the context of an asset market. Intertemporal arbitrage to smooth out fluctuations in asset prices may require holding assets for days or even weeks and here the cost of liquidity may be significantly greater than zero. In the present model, we represent the cost of liquidity by the interest rate charged by the CB on cash balances. If we think of seigniorage as a source of revenue for government expenditures, a positive interest rate is necessary in order to finance a positive level of expenditure. In a more realistic model, the cost of liquidity would be determined not by the rate at which the CB is willing to lend (the discount rate), but rather by the opportunity cost of funds for investment purposes. If arbitrageurs have the opportunity of investing liquid funds in assets that yield a positive return, then any other use of those funds must earn a comparable rate of return. In these circumstances, arbitrage can never be completely successful, because perfect arbitrage eliminates the profit that is needed to justify the use of liquidity for this purpose (see Allen and Gale, 1998, 2002a).

When we discuss the impact of monetary policy on asset prices, it is essential to distinguish *nominal* asset prices from *real* or *relative* asset prices. If we are only interested in the impact of monetary policy on nominal assets prices, the model described in Section 2 is sufficient. We can simply interpret the 'goods' in that model as 'assets' and observe that the general level of asset prices is determined by the CB's policy with respect to money supply and the good.

There are I agents indexed by $i = 1, ..., I$. Each agent i has an endowment of assets $\bar{\theta}_i \in \mathbf{R}^{\ell}_{+}$ and an endowment of goods $e_i \in \mathbf{R}_{+}$. An agent's utility now depends on his portfolio of assets $\theta_i(s)$ and consumption of the good $x_i(s)$ as well as the state s. The state-dependence of preferences allows for liquidity preference shocks. Agent i's preferences are represented by a von Neumann-Morgenstern utility function $u_i : \mathbf{R}^{\ell}_{+} \times \mathbf{R}_{+} \times S \to \mathbf{R}$. We assume that the CB takes its seigniorage in the form of purchases of goods and hence is in a position to control the goods price level, but does not engage in open market operations for assets. Thus, the level of asset prices is determined endogenously, taking as given the money supply and the interest rate determined by the CB.

As before, we take the money supply M as fixed. In the second sub-period, the CB policy demands $g(s) \geq 0$ units of the good in state s and supplies ΔM units of money in exchange.

Note, again, that the amount of money supplied by the CB at date 1 is assumed to be independent of the state s. Market-clearing in the money market is impossible unless $rM = \Delta M$, which implies that ΔM must be independent of s in equilibrium.

4.2 Equilibrium

Agent i borrows $m_i \geq 0$ units of money from the CB in the first sub-period and demands a portfolio $\theta_i(s) \in \mathbf{R}^{\ell}_{+}$ and a quantity of the good $x_i(s) \geq 0$ in each state s in the second sub-period.

Let $q(s) \in \mathbf{R}^{\ell}_{+}$ denote the equilibrium vector of asset prices in state s and let $p(s) \geq 0$ denote the price of the good in state s. Then agent i satisfies the budget constraint

$$q(s) \cdot (\theta_i(s) - \bar{\theta}_i) + p(s)(x_i(s) - e_i) + rm_i \leq 0$$

and the cash-in-advance constraint

$$q(s) \cdot (\theta_i(s) - \bar{\theta}_i)^{+} + p(s)(x_i(s) - e_i)^{+} \leq m_i$$

in each state s in the second sub-period.

An allocation (θ, x, m) is *attainable* if

$$\sum_{i=1}^{I} \theta_i(s) = \sum_{i=1}^{I} \bar{\theta}_i, \forall s,$$

$$\sum_{i=1}^{I} x_i(s) + g(s) = \sum_{i=1}^{I} e_i, \forall s,$$

and

$$\sum_{i=1}^{I} m_i = M.$$

For a given policy $(g, \Delta M)$, an *equilibrium relative to the policy* $(g, \Delta M)$ consists of an attainable allocation (θ, x, m) and a price vector (p, q, r) such that, for every agent i, (θ_i, x_i, m_i) solves

$$\begin{array}{ll} \max & \sum_{s=1}^{S} \pi_s u_i(\theta_i(s), x_i(s), s) \\ \text{s.t} & q(s) \cdot (\theta_i(s) - \bar{\theta}_i) + p(s)(x_i(s) - e_i) + rm_i \leq 0 \\ & q(s) \cdot (\theta_i(s) - \bar{\theta}_i)^+ + p(s)(x_i(s) - e_i)^+ \leq m_i \end{array}$$

and the CB's policy satisfies the budget constraint

$$p(s)g(s) = \Delta M, \forall s.$$

The budget constraints together with the market-clearing conditions imply that $rM = \Delta M$ so there is the right amount of money to pay the loans at the last date.

As before, we can ask what happens as the cost of liquidity r converges to zero. In the limit, the cash-in-advance constraint will have a zero shadow price, but the possibility of variations in the price level is still with us. Since markets are incomplete, variations in the price level, however defined, can have real effects on the equilibrium allocation by varying the value of money held between date 0 and date 1. However, this does not imply that the real price of assets relative to goods (i.e., the price of assets relative to goods) will vary as a result of CB policy. At each date, the usual homogeneity properties continue to hold.

Consider a sequence of policies $(\Delta M^n, g^n) \to (0, 0)$ and a corresponding sequence of equilibria $(a^n, p^n, q^n, r^n) \to (a^0, p^0, q^0, r^0)$, where $a^n = (\theta^n, x^n, m^n)$, $a^n = (\theta^0, x^0, m^0)$ and $r^0 = 0$. In the limit, it is easy to see that $a_i^0 = (\theta_i^0, x_i^0, m_i^0)$ maximizes expected utility $\sum_{s=1}^{S} \pi(s) u(\theta_i(s), x_i(s), s)$ subject to the budget constraints

$$p^0(s) \cdot (x_i(s) - e_i) + q^0(s)(\theta_i(s) - \bar{\theta}_i) \leq 0, \forall s.$$

By the usual argument, the CB policy can affect the goods price level $p^0(s)$ in the limit. Although the CB does not directly impact not the asset prices, because demand is homogeneous of degree zero in $(p^0(s), q^0(s))$, changes in $p^0(s)$ do not necessarily imply a change in the relative prices of assets and goods. A simple example will make this clear. Suppose there is a single asset, $\ell = 1$, and the utility function of agent i is given by

$$u_i(\theta_i(s), x_i(s), s) = \alpha_i(s) \log \theta_i(s) + (1 - \alpha_i(s)) \log x_i(s),$$

where $0 < \alpha_i(s) < 1$. Then, in equilibrium, the demand for the asset is

$$\theta_i(s) = \frac{\alpha_i(s) w_i(s)}{q(s)},$$

where $w_i(s) \equiv p(s)e_i + q(s)\bar{\theta}_i$, and the market-clearing conditions are

$$\sum_{i=1}^{I} \theta_i(s) = \sum_{i=1}^{I} \frac{\alpha_i(s) w_i(s)}{q(s)} = \sum_{i=1}^{I} \bar{\theta}_i$$

or

$$p(s) \sum_{i=1}^{I} \alpha_i(s) e_i = q(s) \sum_{i=1}^{I} (1 - \alpha_i(s)) \bar{\theta}_i.$$

While CB policy can influence the vector $(p(s), q(s))$ the relative price of assets is determined entirely by real factors. Furthermore, it is only *aggregate* demands for assets and goods that affects relative prices. For example, if agents have identical endowments and $\sum_{i=1}^{I} \alpha_i(s)$ is constant across states, then there will be no variation in the relative asset price, even though the volume of trade is variable. The volume of trade depends on the differences in the weights $(\alpha_1(s), ..., \alpha_I(s))$, rather than their sum.

4.3 Costly liquidity

When liquidity is costly, CB policy does impact the asset market in important ways. We continue to use the parametric example introduced above to make some simple points. If $r > 0$, the agent's demand for goods or assets must be liquidity-constrained in some states of nature. For a given agent i and state s there are two possibilities. If

$$\frac{\alpha_i(s) w_i(s)}{q(s)} > \bar{\theta}_i$$

then the agent is either unconstrained or constrained by the quantity of money he holds. In the unconstrained case, the demand for the asset is $\alpha_i(s)w_i(s)/q(s)$. In the constrained case it is $\bar{\theta}_i + m_i/q(s)$. The demand for the asset is given by the minimum of these two expressions

$$\theta_i(s) = \min\left\{\frac{\alpha_i(s)w_i(s)}{q(s)}, \bar{\theta}_i + \frac{m_i}{q(s)}\right\}.$$

The demand for the good, derived from the budget constraint, is

$$x_i(s) = \max\left\{\frac{(1-\alpha_i(s))w_i(s)}{p(s)}, e_i - \frac{m_i}{q(s)}\right\}.$$

In the other case,

$$\frac{\alpha_i(s)w_i(s)}{q(s)} < \bar{\theta}_i$$

the cash-in-advance constraint binds on the demand for the good, if at all. Then

$$\theta_i(s) = \max\left\{\frac{\alpha_i(s)w_i(s)}{q(s)}, \bar{\theta}_i - \frac{m_i}{q(s)}\right\}$$

and

$$\min\left\{\frac{(1-\alpha_i(s))w_i(s)}{p(s)}, e_i + \frac{m_i}{q(s)}\right\}.$$

The important point about this formulation is that liquidity constraints apply equally to both markets. In general, there is no reason to think that the cost of liquidity will affect one market differently from another. As a result, costly liquidity will directly affect the volume of trade in assets but not the asset prices. However, the fact that seigniorage is consumed in the form of purchases of goods rather than assets does introduce an asymmetry which may have an effect on real asset values.

Example 7 *Suppose there are two agents $i = 1, 2$ and two equally probable states $s = 1, 2$. The preference shocks satisfy*

$$\alpha_i(s) = \begin{cases} \alpha_H & \text{if } i = s \\ \alpha_L & \text{if } i \neq s \end{cases}$$

where $0 < \alpha_L < \alpha_H < 1$. The two agents have identical endowments $(\bar{\theta}_i, e_i) = (\bar{\theta}, e)$ for $i = 1, 2$. The government's policy is assumed to satisfy $g(s) = \bar{g}$ for each s and

$$\Delta M = g = rM,$$

so that $p(s) = 1$ for all s. Because of the assumed symmetry of the model, there will exist a symmetric equilibrium in which both agents choose the same money balances $m_i = \bar{m}$ and the asset price $q(s) = \bar{q}$ is a constant. Let (θ_H, x_H) denote the demands of agent i when $\alpha_i(s) = \alpha_H$ and let (θ_L, x_L) denote the demands of agent i when $\alpha_i(s) = \alpha_L$. The market-clearing condition implies that

$$\theta_H - \bar{\theta} = -(\theta_L - \bar{\theta})$$

and the budget constraint implies that

$$-\bar{q}(\theta_L - \bar{\theta}) = (x_L - e) + r\bar{m}.$$

Then the cash-in-advance constraints imply that

$$\bar{m} \geq \bar{q}(\theta_H - \bar{\theta}) = (x_L - e) + r\bar{m}.$$

This implies that the cash-in-advance constraint is binding on type a_H's demand for assets but not on type α_L's demand for goods (since the interest payment is made in the third sub-period so the cash-in-advance constraint has the form $(x_L - e) < \bar{m}$). An increase in r will reduce the demand for $\bar{m}$ and the demand for the asset from type α_H with the result that the equilibrium value of $\bar{q}$ falls.

The asymmetric pressure of the liquidity constraint on the asset market arises from the fact that the cash-in-advance constraint is only binding in the asset market. This depends on two features of the model: (a) interest payments are made in the final sub-period and therefore do not add to the demand for liquidity in the middle sub-period and (b) the CB is purchasing goods so the volume of (private) trade in assets is greater than the volume of (private) trade in goods.

4.4 Market separation

This suggests a channel by which liquidity can affect volatility. Suppose that goods and assets are traded on separate markets. An agent has to satisfy an independent cash-in-advance constraint in each market, one for assets and one for goods. Because of the independent cash-in-advance constraints, a liquidity shock in the asset market will not have a direct effect on the goods market and hence on the goods price level. For example, if the volume of trades in the asset market is unusually high, the demand for assets is constrained by a shortage of liquidity but the supply is not and a temporary excess supply will depress asset prices. Goods prices are not directly affected because the demand for and supply of liquidity in the goods market are unaffected. This allows for the possibility that nominal asset prices can fluctuate independently of nominal goods prices, with the result that monetary policy can change real asset prices measured in terms of goods. By contrast, if all trades were aggregated in a single cash-in-advance constraint, a liquidity shock from whatever source would effect the prices of goods and assets similarly.

In the rest of this section, we extend the description of equilibrium from Section 4.1 to deal with segregated trading of assets and goods and then show how costly liquidity influences the pricing of assets.

4.5 Equilibrium

The market structure now consists of four sub-periods. In the first, agents borrow money from the CB. In the second, they trade assets for money (without any intervention by the CB). In the third, the agents and the CB exchange goods for money. In the fourth sub-period, agents repay their loans with interest. Agent i borrows $m_i \geq 0$ units of money from the CB in the first sub-period. In the second sub-period he observes the state of nature s and demands a portfolio of assets $\theta_i(s) \in \mathbf{R}^{\ell}_{+}$. In the third sub-period, he demands a quantity of the good $x_i(s) \geq 0$ in each state s.

Let $q(s) \in \mathbf{R}^{\ell}_{+}$ denote the equilibrium vector of asset prices in state s and let $p(s) \geq 0$ denote the price of the good in state s. Then agent i satisfies the budget constraint

$$q(s) \cdot (\theta_i(s) - \bar{\theta}_i) + p(s)(x_i(s) - e_i) + rm_i \leq 0$$

and the cash-in-advance constraint

$$q(s) \cdot (\theta_i(s) - \bar{\theta}_i)^+ \leq m_i$$

in each state s in the second sub-period. Since there is only one good traded for money in the third sub-period, the budget constraint implies that the cash-in-advance constraint is satisfied in that sub-period.

The definition of an attainable allocation (θ, x, m) is the same as the one used in Section 4.2. We revise the definition of equilibrium given in

Section 4.2 as follows. For a given policy $(g, \Delta M)$, an *equilibrium relative to the policy* $(g, \Delta M)$ consists of an attainable allocation (θ, x, m) and a price vector (p, q, r) such that, for every agent i, (θ_i, x_i, m_i) solves

$$\begin{array}{ll} \max & \sum_{s=1}^{S} \pi_s u_i(\theta_i(s), x_i(s), s) \\ \text{s.t} & p(s)(x_i(s) - e_i) + q(s) \cdot (\theta_i(s) - \bar{\theta}_i) + r m_i \leq 0 \\ & q(s) \cdot (\theta_i(s) - \bar{\theta}_i)^+ \leq m_i \end{array}$$

and the CB's policy satisfies the budget constraint

$$p(s)g(s) = \Delta M, \forall s.$$

As usual, the budget constraints together with the market-clearing conditions imply that $rM = \Delta M$, so there is the right amount of money to pay the loans at the last date.

The separation of the two markets has no effect as long as the cost of liquidity is vanishingly small. To see this, we let r converge to zero. Consider a sequence of policies $(\Delta M^n, g^n) \to (0, 0)$ and a corresponding sequence of equilibria $(a^n, p^n, q^n, r^n) \to (a^0, p^0, q^0, r^0)$, where $a^n = (\theta^n, x^n, m^n)$, $a^n = (\theta^0, x^0, m^0)$ and $r^0 = 0$. In the limit, it is easy to see that $a_i^0 = (\theta_i^0, x_i^0, m_i^0)$ maximizes expected utility $\sum_{s=1}^{S} \pi(s) u(\theta_i(s), x_i(s), s)$ subject to the budget constraints

$$p^0(s) \cdot (x_i(s) - e_i) + q^0(s)(\theta_i(s) - \bar{\theta}_i) \leq 0, \forall s.$$

In the limit, the cash-in-advance constraint will have a zero shadow price, and agent's choices are not liquidity-constrained. The CB can control the price level $p^0(s)$ so the possibility of variations in the price level is still with us but the usual homogeneity properties continue to hold and so the CB's policy has no effect on the equilibrium set.

4.6 Liquidity shocks and asset-price volatility

To see how equilibrium changes when the cost of liquidity is positive suppose that $(g, \Delta M) \neq 0$ so that $r > 0$ in equilibrium. The calculation of equilibrium values is complicated by the fact that asset demands are cash-constrained, so we revert to our parametric example and assume that there is a single asset $(\ell = 1)$ agents have log-linear preferences:

$$u_i(\theta_i(s), x_i(s), s) = \alpha_i(s) \log \theta_i(s) + (1 - \alpha_i(s)) \log x_i(s).$$

If we let $w_i(s) \equiv p(s)e_i + q(s)\bar{\theta}_i - rm_i$, then optimality and the cash-in-advance constraint imply that

$$\theta_i(s) = \min\left\{\frac{\alpha_i(s)w_i(s)}{q(s)}, \frac{m_i}{q(s)} + \bar{\theta}_i\right\}$$

and the budget constraint implies that

$$\begin{aligned} x_i(s) &= \frac{w_i(s) - q(s)\left(\theta_i(s) - \bar{\theta}_i\right)}{p(s)} \\ &= \max\left\{\frac{(1-\alpha_i(s))w_i(s)}{p(s)}, e_i - \frac{(1+r)m_i}{p(s)}\right\}. \end{aligned}$$

Then the agent's utility can be expressed as

$$\sum_s \pi(s)\left\{\alpha_i(s)\ln\left[\min\left\{\frac{\alpha_i(s)w_i(s)}{q(s)}, \frac{m_i}{q(s)} + \bar{\theta}_i\right\}\right] + \right.$$
$$\left.(1-\alpha_i(s))\ln\left[\max\left\{\frac{(1-\alpha_i(s))w_i(s)}{p(s)}, e_i - \frac{(1-r)m_i}{p(s)}\right\}\right]\right\}.$$

To illustrate the role of liquidity in asset-pricing volatility (Allen-Gale, 1994), consider the special case in which there is no aggregate uncertainty about the demand for the asset but there is uncertainty about the individual demands for the asset. In a model with complete markets and no cost of liquidity, asset prices would be constant across states. However, the existence of costly liquidity means that the value of trades will be constrained by the liquidity held by the agents in the market. When volume of trade is high, the value of trades is constrained by the liquidity in the market. The only way this can be satisfied in equilibrium is if asset prices fall when the volume of trade is high. Thus, asset prices are determined by the amount of "cash in the market," rather than by primitive parameters such as asset returns and agents' risk aversion and discount rates.

Formally, we assume that

- agents are ex ante identical, that is, they have the same endowments

  $$(e_i, \bar{\theta}_i) = (e, \bar{\theta}), \forall i$$

  and their preference parameters $\alpha_i(s)$ have identical probability distributions, and

- aggregate demand for assets is constant, that is, there exists a constant $\bar{\alpha}$ such that

$$I^{-1}\sum_i \alpha(s) = \bar{\alpha}, \forall s.$$

We consider a symmetric equilibrium in which the prices (p, q, r) depend only on the aggregate distribution of parameter shocks $\{\alpha_i(s)\}$. Since all agents are ex ante identical, their choices in the first sub-period will be the same and this implies that $m_i = m$ for each i. Substituting these expressions in the demand functions derived earlier gives us

$$\theta_i(s) = \min\left\{\frac{\alpha_i(s)\hat{w}(s)}{q(s)}, \frac{\hat{b}}{q(s)} + \hat{\theta}\right\}$$

and

$$x_i(s) = \max\left\{\frac{(1-\alpha_i(s))\hat{w}(s)}{p(s)}, \hat{e} - \frac{(1-r)\hat{b}}{p(s)}\right\},$$

where $\hat{w}(s) \equiv p(s)\hat{e} + q(s)\hat{\theta} - r\hat{b}$.

For the sake of illustration, suppose that the CB's policy is to maintain $p(s) = 1$. If the demand for assets is not constrained by money balances at date 1 then the aggregate demand for assets in state s is

$$\begin{aligned}
\sum_i \theta_i(s) &= \sum_i \min\left\{\frac{\alpha_i(s)w(s)}{q(s)}, \frac{m}{q(s)} + \bar{\theta}\right\} \\
&= \sum_i \frac{\alpha_i(s)w(s)}{q(s)} \\
&= \frac{\bar{\alpha}I(e + q(s)\bar{\theta})}{q(s)}
\end{aligned}$$

where we assume that $r = 0$ (otherwise the constraint would be binding). The market-clearing condition is

$$I\bar{\theta} = \frac{\bar{\alpha}I(e + q(s)\bar{\theta})}{q(s)} = \bar{\alpha}I\left(\bar{\theta} + \frac{e}{q(s)}\right).$$

This equation has a unique solution $q(s) = \bar{q}$ independently of s. Thus, absent liquidity constraints, there is no asset price volatility. This is what will happen in the limit as $r \searrow 0$. However, if liquidity is costly, we have already seen that it pays to economize on liquidity. In other words, at the unconstrained optimum, a small reduction in the demand for assets has no first-order impact on utility but does have a first-order impact on liquidity costs. Thus, in some states, we expect that demand is constrained by cash balances for large values of $\alpha_i(s)$. Let $\hat{I}(s)$ denote the set of unconstrained agents in state s. Then the market-clearing condition is

$$\begin{aligned} I\bar{\theta} &= \sum_i \min\left\{\frac{\alpha_i(s)w(s)}{q(s)}, \frac{m}{q(s)} + \bar{\theta}\right\} \\ &= \sum_{i \in \hat{I}(s)} \frac{\alpha_i(s)w(s)}{q(s)} + \sum_{i \notin \hat{I}(s)} \frac{m + q(s)\hat{\theta}}{q(s)} \\ &< \sum_i \frac{\alpha_i(s)w(s)}{q(s)} \\ &= \bar{\alpha} I \left(\bar{\theta} + \frac{e - rm}{q(s)}\right). \end{aligned}$$

Comparing this inequality with the earlier market-clearing condition we see that the solution in the constrained case satisfies $q(s) < \bar{q}$.

In fact, the more agents are constrained the lower the asset price. Consider two states s and s'. If $\{\alpha_i(s)\}$ is a mean preserving spread of $\{\alpha_i(s')\}$ then $q(s) \leq q(s')$.

In this exercise, we have assumed that the CB's policy is to keep the price of goods constant across states, so that seigniorage is constant. This implies that changes in the nominal price of assets are equivalent to changes in the relative price of assets, which is what we are interested in. Is it possible that by manipulating the price level, the real asset price could be stabilized? The answer is yes, but only by varying the real value of seigniorage across states. That is, making the price level $p(s)$ low when $q(s)$ is low means that the real value of the interest charge $rm/p(s)$ is higher. There will be a tradeoff between the value of stabilizing the asset price and destabilizing the real value of seigniorage.

5 Discussion

[To be completed]

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